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Shape Preserving Properties of Some Positive Linear Operators on Unbounded Intervals*

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In this paper we study some shape preserving properties of particular positive linear operators acting on spaces of continuous functions defined on the interval $[0, +\infty[$, which are strongly related to the semigroups generated by a large class of degenerate elliptic second order differential operators. We study the conditions under which these operators leave invariant the class of increasing functions, as well as the class of convex functions and Hölder continuous functions. As a consequence, we derive some regularity results concerning the related semigroups.

INTRODUCTION

Given a Banach space E and a closed linear operator A: $D(A) \to E$ which generates a C_0-semigroup $(T(t))_{t \geqslant 0}$ on E, it is well known that the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geqslant 0 \\ u(0) = u_0, & u_0 \in D(A), \end{cases} \tag{I}$$

has a unique solution u: $[0, +\infty[\to E$ given by

$$u(t) = T(t)\, u_0 \qquad (t \geqslant 0). \tag{II}$$

Since the explicit expression of the semigroup $(T(t))_{t \geqslant 0}$ is generally unknown, in order to investigate qualitative properties of the solution (II)

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of problem (I), during these last years F. Altomare has developed a useful and elegant technique which consists in constructing suitable approximation processes $(T_n)_{n \geqslant 1}$ on the space E whose powers converge strongly to the given semigroup, that is,

$$\lim_{n \to \infty} T_n^{k(n)} f = T(t) f \qquad (t \geqslant 0, f \in E) \tag{III}$$

for some sequence $(k(n))_{n \geqslant 1}$ of positive integer numbers.

Having the formula (III) at our disposal, some regularity results for problem (I) can be obtained by investigating those closed subsets of E which are invariant under T_n $(n \geqslant 1)$.

In the setting of Banach spaces of continuous functions on bounded intervals, this problem has been tackled by several mathematicians (see, e.g., [1]) in the case of A being an elliptic second-order differential operator.

In two recent papers [3, 4] the case of unbounded intervals has been also considered. It was introduced a sequence of positive linear operators acting on weighted function spaces on the interval $[0, +\infty[$, denoted by $(M_{n,\lambda})_{n \geqslant 1}$, where λ is a continuous function on $[0, +\infty[$ such that $0 \leqslant \lambda \leqslant 1$.

Furthermore, generalizing some results obtained by Becker and Nessel [5], it was shown that the differential operator $A_\lambda u(x) := (x\lambda(x)/2)\, u''(x)$ $(x > 0)$ defined on a suitable domain of W_2^0, generates a C_0-semigroup $(T_\lambda(t))_{t \geqslant 0}$ which can be expressed in terms of the sequence $(M_{n,\lambda})_{n \geqslant 1}$, via formula (III) [4]. Here W_2^0 denotes the space of all continuous functions f on $[0, +\infty[$ satisfying $\lim_{x \to +\infty}(f(x)/(1+x^2)) = 0$.

In the present paper we intend to study some qualitative properties of the sequence $(M_{n,\lambda})_{n \geqslant 1}$, in light of the above-mentioned introduction.

In fact, we shall prove that each operator $M_{n,\lambda}$ preserves monotonicity, convexity, and convexity of all orders, provided the function λ is constant. By a simple counterexample we shall also show that the above results are not true if λ is not constant.

Finally, we shall present a result concerning the preservation of classes of Hölder continuous functions and, moreover, we shall prove that each $M_{n,\lambda}$ leaves invariant the class of Lipschitz functions if and only if λ is constant.

1. NOTATION AND MAIN DEFINITIONS

In this section we recall the definitions and the main properties of the positive linear operators whose shape preserving properties we shall deal with. For more details about what follows we refer the reader to [3, 4].

We shall denote by $C([0, +\infty[)$ the vector space of all real-valued continuous functions on $[0, +\infty[$ and we shall consider the Banach lattice $C_b([0, +\infty[)$ of all real-valued bounded continuous functions on $[0, +\infty[$ endowed with the natural order and the sup-norm

$$\|f\|_\infty := \sup_{x \geqslant 0} |f(x)| \qquad (f \in C_b([0, +\infty[)). \tag{1.1}$$

Moreover, we shall denote by $C^*([0, +\infty[)$ the subspace of all functions $f \in C_b([0, +\infty[)$ which are convergent at infinity.

For every $\alpha > 0$ the symbol E_α will indicate the subspace of all $f \in C([0, +\infty[)$ such that $\sup_{x \geqslant 0}(|f(x)|/e^{\alpha x}) < +\infty$. The space E_α endowed with the natural order and the norm

$$\|f\|_\alpha := \sup_{x \geqslant 0} \frac{|f(x)|}{e^{\alpha x}} \qquad (f \in E_\alpha), \tag{1.2}$$

becomes a Banach lattice.

We shall also set

$$E_\infty := \bigcup_{\alpha > 0} E_\alpha. \tag{1.3}$$

Throughout this paper we shall fix a function $\lambda \in C_b([0, +\infty[)$ satisfying $0 \leqslant \lambda(x) \leqslant 1$ for every $x \in [0, +\infty[$.

For such a λ and for each $x \in [0, +\infty[$, we consider the distribution

$$\rho_{x, \lambda} := \lambda(x)\, \pi_x + (1 - \lambda(x))\, \varepsilon_x, \tag{1.4}$$

where ε_x and π_x denote the unit mass at x and the Poisson distribution on $\mathbb{R}$ with parameter x, i.e., $\pi_x := \sum_{k=0}^{\infty} e^{-x}(x^k/k!)\, \varepsilon_k$ (with the convention $\pi_0 := \varepsilon_0$).

For every $n \in \mathbb{N}$, $n \geqslant 1$, each operator $M_{n, \lambda}\colon E_\infty \to E_\infty$ is defined by

$$M_{n, \lambda}(f)(x) := \int_0^{+\infty} \cdots \int_0^{+\infty} f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\rho_{x, \lambda}(x_1) \cdots d\rho_{x, \lambda}(x_n) \tag{1.5}$$

for all $f \in E_\infty$ and $x \in [0, +\infty[$.

In [3] an explicit expression of the above operators is also given, namely,

$$M_{n, \lambda}(f)(x) = \sum_{p=0}^{n} \sum_{h=0}^{\infty} \binom{n}{p} \lambda(x)^p\, (1 - \lambda(x))^{n-p}\, e^{-px} \frac{(px)^h}{h!}\, f\left(\frac{h}{n} + \left(1 - \frac{p}{n}\right) x\right) \tag{1.6}$$

for every $f \in E_\infty$, $x \geqslant 0$ and $n \in \mathbb{N}$, $n \geqslant 1$.

The operators $M_{n,\lambda}$ $(n \geqslant 1)$ can be considered as a generalization of the well-known Szàsz–Mirakjan operators M_n, defined by

$$M_n(f)(x) := \sum_{h=0}^{\infty} e^{-nx} \frac{(nx)^h}{h!} f\left(\frac{h}{n}\right) \qquad (f \in E_\infty). \tag{1.7}$$

In fact, if we consider the function $f_{n,p,x}\colon [0, +\infty[\to \mathbb{R}$ defined by

$$f_{n,p,x}(t) := f\left(\frac{p}{n} t + \left(1 - \frac{p}{n}\right) x\right) \qquad (t \geqslant 0) \tag{1.8}$$

for fixed $x \geqslant 0$, $n \geqslant 1$, and $p \in \{0, 1, 2, ..., n\}$, then the following representation of the operators $M_{n,\lambda}$ $(n \geqslant 1)$ in terms of the operators M_n holds true:

$$M_{n,\lambda}(f)(x) = \sum_{p=0}^{n} \binom{n}{p} \lambda(x)^p (1 - \lambda(x))^{n-p} M_p(f_{n,p,x})(x), \tag{1.9}$$

with the convention $M_0(f_{n,p,x})(x) := f(x)$.

Note that, when $\lambda = \mathbf{1}$, i.e., $\lambda(x) = 1$ for all $x \in [0, +\infty[$, then $M_{n,\mathbf{1}}$ becomes the nth Szàsz–Mirakjan operator M_n, that is

$$M_{n,\mathbf{1}}(f)(x) = M_n(f)(x). \tag{1.10}$$

We shall point out that a probabilistic interpretation of the above operators can be given (see [2, Sect. 5.2]), so that the operators $M_{n,\lambda}$ are, in fact, Feller-type operators.

More precisely, let us consider a probability space (Ω, F, P) and an independent family $(Y(n, x))_{n \geqslant 1, x \geqslant 0}$ of real random variables on Ω satisfying the condition

$$P_{Y(n,x)} = \rho_{x,\lambda} \tag{1.11}$$

for all $n \geqslant 1$ and $x \geqslant 0$, where $P_{Y(n,x)}$ denotes the distribution of $Y(n, x)$. For all $n \geqslant 1$ and $x \geqslant 0$, we also consider the random variable

$$X(n, x) := \frac{1}{n} \sum_{k=1}^{n} Y(k, x) \tag{1.12}$$

which takes its values in $[0, +\infty[$ P-almost surely.

Extending the technique used in [3, Sect. 1], one easily obtains that for every $f \in E_\infty$, $n \geqslant 1$ and $x \geqslant 0$

$$M_{n,\lambda}(f)(x) = \int_0^{+\infty} f \, dP_{X(n,x)}. \tag{1.13}$$

This formula is a generalization of formula (1.9) of the above-mentioned paper [3].

In our investigations we shall also consider the weighted space W_2^0 consisting of all functions $f \in C([0, +\infty[)$ such that

$$f(x) = o\left(\frac{1}{w_2(x)}\right), \qquad x \to +\infty, \tag{1.14}$$

where w_2 denotes the weight function

$$w_2(x) := \frac{1}{1+x^2} \qquad (x \geqslant 0). \tag{1.15}$$

The space W_2^0, endowed with the weighted norm

$$\|f\|_2 := \sup_{x \geqslant 0} w_2(x) \, |f(x)| \qquad (f \in W_2^0), \tag{1.16}$$

becomes a Banach space.

In this setting we consider the differential operator $A_\lambda : D(A_\lambda) \to W_2^0$ defined by

$$A_\lambda u(x) := \begin{cases} \dfrac{x\lambda(x)}{2} u''(x), & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \end{cases} \tag{1.17}$$

for all $u \in D(A_\lambda)$, where the domain $D(A_\lambda)$ of A_λ is the subspace of all functions $u \in W_2^0 \cap C^2(]0, +\infty[)$ which satisfy the Wentcel's boundary conditions

$$\lim_{x \to 0^+} xu''(x) = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} w_2(x) \, xu''(x) = 0. \tag{1.18}$$

In [4, Corollary 3.3]) it was proved that, under the assumption

$$\lambda(x) \geqslant \lambda_0 > 0 \qquad (x \geqslant 0), \tag{1.19}$$

the operator $(A_\lambda, D(A_\lambda))$ defined by (1.17) and (1.18) generates a C_0-semigroup $(T_\lambda(t))_{t \geqslant 0}$ on the space W_2^0.

Moreover, if for every $k \in \mathbb{N}$, $k \geqslant 1$, we denote by $M_{n,\lambda}^k$ the power of order k of the operator $M_{n,\lambda}$, i.e.,

$$M_{n,\lambda}^k := \begin{cases} M_{n,\lambda}, & \text{if} \quad k = 1, \\ M_{n,\lambda} \circ M_{n,\lambda}^{k-1}, & \text{if} \quad k \geqslant 2, \end{cases} \tag{1.20}$$

then the semigroup $(T_\lambda(t))_{t\geqslant 0}$ can be represented in terms of the above powers in the following way (see [4, Theorem 4.3]):

$$T_\lambda(t)\, f = \lim_{n\to\infty} M_{n,\lambda}^{k(n)} f \qquad \text{in} \quad W_2^0 \tag{1.21}$$

for every $f \in W_2^0$ and $t \geqslant 0$ and for every sequence $(k(n))_{n\geqslant 1}$ of positive integers such that $\lim_{n\to\infty}(k(n)/n) = t$.

In particular, each operator $T_\lambda(t)$ is positive and formula (1.21) also holds uniformly on compact subsets of $[0, +\infty[$.

Furthermore, note that the Cauchy problem

$$\begin{cases} \dot{u}(t) = A_\lambda u(t), & (t > 0), \\ u(0) = u_0, & u_0 \in D(A_\lambda), \end{cases} \tag{1.22}$$

has a unique solution given by

$$u(t) = T_\lambda(t)\, u_0 \qquad (t \geqslant 0). \tag{1.23}$$

So, by means of formula (1.21), shape preserving properties of operators $M_{n,\lambda}$ can be transferred to the operators $T_\lambda(t)$ and hence to the solution of problem (1.22).

In fact, this was the main motivation of this paper and the results we presented here represent a first step towards a more comprehensive analysis that will be carried out in the sequel.

2. MONOTONICITY AND CONVEXITY

In this section we present some results concerning monotonicity and convexity in the spaces E_∞, $C_b([0, +\infty[)$ and W_2^0.

We start with a theorem which gives several characterizations of a convex function in the space W_2^0.

THEOREM 2.1. *If* $f \in W_2^0$ *and if* λ *satisfies* (1.19), *then the following statements are equivalent*:

(a) f *is convex*;

(b) $M_{n+1,\lambda}(f) \leqslant M_{n,\lambda}(f)$ *for every* $n \geqslant 1$;

(c) $f \leqslant M_{n,\lambda}(f)$ *for every* $n \geqslant 1$;

(d) $f \leqslant T_\lambda(t)\, f$ *for every* $t \geqslant 0$;

(e) $T_\lambda(t)\, f$ *is convex for every* $t \geqslant 0$.

Proof. (a) ⇒ (b): Part (b) easily follows from representation formula (1.13) by using the same reasoning as in the proof of Theorem 3 of [8]. There the proof is based on the conditional version of Jensen's inequality which also holds true for integrable convex functions (see, e.g., [6, 10.2.7]).

(b) ⇒ (c): One easily gets

$$M_{n+m,\lambda}(f) \leqslant M_{n,\lambda}(f) \qquad (n, m \geqslant 1, x \geqslant 0). \tag{1}$$

On the other hand, from Corollary 2.5 of [4] we know that

$$\lim_{n \to \infty} M_{n,\lambda}(f)(x) = f(x) \qquad (x \geqslant 0).$$

Consequently, part (c) follows letting $m \to \infty$ in (1).

(c) ⇒ (d): Suppose that $f \leqslant M_{n,\lambda}(f)$ for every $n \geqslant 1$. Then $f \leqslant M^{n}_{n,\lambda}(f)$ for every $n, m \geqslant 1$. In particular, if $(k(n))_{n \geqslant 1}$ is a sequence of positive integers such that $\lim_{n \to \infty}(k(n)/n) = t$, then

$$f(x) \leqslant M^{k(n)}_{n,\lambda}(f)(x) \qquad (n \geqslant 1, x \geqslant 0). \tag{2}$$

Letting $n \to \infty$ in (2) and taking formula (1.21) into account, we obtain part (d).

(d) ⇒ (a): Suppose that $f \leqslant T_\lambda(t) f$ for every $t \geqslant 0$.

Since $\int_0^t T_\lambda(s) f\, ds \in D(A_\lambda)$ and $A_\lambda(\int_0^t T_\lambda(s) f\, ds) = T_\lambda(t) f - f \geqslant 0$ for every $t \geqslant 0$ (see, e.g., [2, Theorem 1.6.1(5)]), then the function $\int_0^t T_\lambda(s) f\, ds$ is convex $(t \geqslant 0)$. Accordingly, $f = \lim_{t \to 0^+} \int_0^t T_\lambda(s) f\, ds$ is convex too.

(a) ⇒ (e): Suppose that f is convex. From (d) and the positivity of the semigroup $(T_\lambda(t))_{t \geqslant 0}$ it follows that

$$T_\lambda(s) f \leqslant T_\lambda(s)(T_\lambda(t) f) = T_\lambda(t)(T_\lambda(s) f) \qquad (s, t \geqslant 0).$$

Thus $T_\lambda(s) f$ is convex for all $s \geqslant 0$ because the implication (d) ⇒ (a) holds true for the function $T_\lambda(s) f$ as well.

(e) ⇒ (a): If $T_\lambda(t) f$ is convex for all $t \geqslant 0$, in particular we obtain that $f = T_\lambda(0) f$ is convex. ∎

Observe that $M_{n,\lambda}(f)(0) = f(0)$ for all $f \in E_\infty$ and $n \geqslant 1$. Moreover, if $f \in W^0_2$ and the function λ satisfies assumption (1.19), then $T_\lambda(t) f(0) = f(0)$ $(t \geqslant 0)$.

Now we present other results in which we shall suppose that the function λ is constant. We shall also give a counterexample (Remark 2.4) in order to show that these results do not hold true without this hypothesis.

In the following proposition we shall see that each operator $M_{n,\lambda}$ maps increasing functions into increasing functions.

PROPOSITION 2.2. *Suppose that λ is constant. Then a function $f \in E_\infty$ is increasing if and only if $M_{n,\lambda}(f)$ is increasing for each $n \geqslant 1$.*

Moreover, if $f \in W_2^0$ is increasing, then $T_\lambda(t) f$ is increasing for every $t \geqslant 0$.

Proof. Let $f \in E_\infty$ be an increasing function and fix $n \geqslant 1$ and $x, y \in \mathbb{R}$ such that $0 \leqslant x < y$.

We preliminary note that, for every $p = 0, 1, ..., n$, from the monotonicity of f and the definition (1.8) of $f_{n,p,x}$ it follows that

$$f_{n,p,x}(t) \leqslant f_{n,p,y}(t) \qquad (t \geqslant 0). \tag{1}$$

From (1) and from the positivity of Szàsz–Mirakjan operators we obtain

$$M_p(f_{n,p,x}) \leqslant M_p(f_{n,p,y}). \tag{2}$$

On the other hand, it is easy to see that for every $p = 0, 1, ..., n$, the function $f_{n,p,x}$ is increasing, i.e.,

$$f_{n,p,x}(u) \leqslant f_{n,p,x}(v) \qquad (0 \leqslant u < v < +\infty). \tag{3}$$

Finally, taking formulas (2), (3) and (1.9) into account and using the fact that each operator M_p preserves monotonicity (see [9, Theorem I, formula (8) and Theorem II]), we get

$$\begin{aligned} M_{n,\lambda}(f)(x) &= \sum_{p=0}^{n} \binom{n}{p} \lambda^p (1-\lambda)^{n-p} M_p(f_{n,p,x})(x) \\ &\leqslant \sum_{p=0}^{n} \binom{n}{p} \lambda^p (1-\lambda)^{n-p} M_p(f_{n,p,y})(x) \\ &\leqslant \sum_{p=0}^{n} \binom{n}{p} \lambda^p (1-\lambda)^{n-p} M_p(f_{n,p,y})(y) = M_{n,\lambda}(f)(y). \end{aligned}$$

Hence, $M_{n,\lambda}(f)$ is increasing.

Conversely, if $M_{n,\lambda}(f)$ is increasing for every $n \geqslant 1$, then f is increasing too, since $\lim_{n\to\infty} M_{n,\lambda}(f)(x) = f(x)$ for all $f \in E_\infty$ and $x \geqslant 0$ (see [3, Theorem 2.3(1)]).

In particular, if $f \in W_2^0$, then the assertion directly follows from the above proof and the representation formula (1.21). ∎

As regards the preservation of convexity, we shall show the following result.

THEOREM 2.3. *Suppose that λ is constant. Then a function $f \in E_\infty$ is convex if and only if $M_{n,\lambda}(f)$ is convex for each $n \geqslant 1$.*

Proof. We first prove the direct implication.

Let us fix a convex function $f \in E_\infty$, $x, y \in \mathbb{R}$ such that $0 \leqslant x < y$, $n \geqslant 1$ and $p = 0, 1, ..., n$.

We shall divide the proof into several steps.

We begin by showing that

$$f_{n,p,z} \text{ is convex on } [0, +\infty[\text{ for all } z \geqslant 0. \tag{1}$$

In fact, since f is convex, for every $u, v, z \in [0, +\infty[$ and $t \in [0, 1]$ one has

$$\begin{aligned} f_{n,p,z}(tu + (1-t)v) &= f\left(t\left[\frac{p}{n}u + \left(1 - \frac{p}{n}\right)z\right] + (1-t)\left[\frac{p}{n}v + \left(1 - \frac{p}{n}\right)z\right]\right) \\ &\leqslant tf\left(\frac{p}{n}u + \left(1 - \frac{p}{n}\right)z\right) + (1-t)\, f\left(\frac{p}{n}v + \left(1 - \frac{p}{n}\right)z\right) \\ &= tf_{n,p,z}(u) + (1-t)\, f_{n,p,z}(v). \end{aligned}$$

Moreover, for every $s, t \in [0, +\infty[$ such that $s < t$, we have

$$(f_{n,p,y} - f_{n,p,x})(s) \leqslant (f_{n,p,y} - f_{n,p,x})(t). \tag{2}$$

Indeed, if we set

$$a := \frac{p}{n}s + \left(1 - \frac{p}{n}\right)x, \qquad b := \frac{p}{n}s + \left(1 - \frac{p}{n}\right)y,$$

$$c := \frac{p}{n}t + \left(1 - \frac{p}{n}\right)x, \qquad d := \frac{p}{n}t + \left(1 - \frac{p}{n}\right)y,$$

since $b, c \in [a, d]$, $b + c = a + d$ and $0 \leqslant a \leqslant d$, then from the convexity of f it follows that $f(b) + f(c) \leqslant f(a) + f(d)$, i.e.,

$$f_{n,p,y}(s) + f_{n,p,x}(t) \leqslant f_{n,p,x}(s) + f_{n,p,y}(t),$$

which implies (2).

Since each operator M_p preserves monotonicity, we obtain that $M_p(f_{n,p,y}) - M_p(f_{n,p,x})$ is increasing and hence

$$M_p(f_{n,p,x})(y) + M_p(f_{n,p,y})(x) \leqslant M_p(f_{n,p,x})(x) + M_p(f_{n,p,y})(y). \tag{3}$$

Furthermore, a simple calculation gives

$$f_{n,\,p,\,tu+(1-t)\,v}(z) \leqslant t f_{n,\,p,\,u}(z) + (1-t)\, f_{n,\,p,\,v}(z) \tag{4}$$

for every $u, v, z \geqslant 0$ and $t \in [0, 1]$.

In particular, from (4) it follows that

$$f_{n,\,p,\,(x+y)/2}(z) \leqslant \tfrac{1}{2}(f_{n,\,p,\,x}(z) + f_{n,\,p,\,y}(z)) \qquad (z \geqslant 0);$$

hence, using the positivity of each operator M_p we get

$$M_p\,(f_{n,\,p,\,(x+y)/2})\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}\left(M_p(f_{n,\,p,\,x})\left(\frac{x+y}{2}\right) + M_p(f_{n,\,p,\,y})\left(\frac{x+y}{2}\right)\right). \tag{5}$$

Since each M_p preserves convexity (see [9, Theorem I, formula (8) and Theorem II]), from (1), (3) and (5) it follows that

$$\begin{aligned} &M_p(f_{n,\,p,\,(x+y)/2})\left(\frac{x+y}{2}\right) \\ &\quad\leqslant \frac{1}{4}\,(M_p(f_{n,\,p,\,x})(x) + M_p(f_{n,\,p,\,x})(y) + M_p(f_{n,\,p,\,y})(x) + M_p(f_{n,\,p,\,y})(y)) \\ &\quad\leqslant \frac{1}{2}\,(M_p(f_{n,\,p,\,x})(x) + M_p(f_{n,\,p,\,y})(y)). \end{aligned} \tag{6}$$

Thus the function $z \mapsto M_p(f_{n,\,p,\,z})(z)$ is convex on $[0, +\infty[$, since it is continuous and (6) holds.

Consequently, for every $u, v \geqslant 0$ and $t \in [0, 1]$, one has

$$\begin{aligned} &M_{n,\,\lambda}(f)(tu + (1-t)\,v) \\ &\quad\leqslant \sum_{p=0}^{n} \binom{n}{p} \lambda^p (1-\lambda)^{n-p}\,(tM_p(f_{n,\,p,\,u})(u) + (1-t)\, M_p(f_{n,\,p,\,v})(v)) \\ &\quad= tM_{n,\,\lambda}(f)(u) + (1-t)\, M_{n,\,\lambda}(f)(v). \end{aligned}$$

So, $M_{n,\,\lambda}(f)$ is convex on $[0, +\infty[$.

Conversely, if $M_{n,\,\lambda}(f)$ is convex for every $n \geqslant 1$, then f is convex too since $\lim_{n\to\infty} M_{n,\,\lambda}(f)(x) = f(x)$ for all $f \in E_\infty$ and $x \geqslant 0$ (see [3, Theorem 2.3(1)]). ∎

Remark 2.4. If the function λ is not constant, then in general on the space E_∞ the operators $M_{n,\,\lambda}$ do not map convex (resp. increasing) functions into convex (resp., increasing) functions.

For this purpose, we present a simple counterexample.

Let us consider the function e_2 defined by $e_2(x) := x^2$ $(x \geqslant 0)$. Clearly, $e_2 \in E_\infty$ and is convex and increasing on $[0, +\infty[$.

It is very easy to check that if we choose as λ the function

$$\lambda(x) := \begin{cases} \alpha x - \beta x^2, & \text{if} \quad 0 \leqslant x \leqslant \dfrac{\alpha}{\beta}, \\ 0, & \text{if} \quad x \geqslant \dfrac{\alpha}{\beta}, \end{cases}$$

with $\alpha > 2$ and $\beta \geqslant \alpha^2/4$, then $M_{1, \lambda}(e_2) = \lambda e_1 + e_2$ (cf. [4, Lemma 4.1]), i.e.,

$$M_{1, \lambda}(e_2)(x) = \begin{cases} (\alpha + 1)\, x^2 - \beta x^3, & \text{if} \quad 0 \leqslant x \leqslant \dfrac{\alpha}{\beta}, \\ x^2, & \text{if} \quad x \geqslant \dfrac{\alpha}{\beta}. \end{cases}$$

Accordingly, $M_{1, \lambda}(e_2)$ is neither everywhere convex nor increasing on $[0, +\infty[$.

We end this section by presenting a result concerning the class of convex functions of order greater than 2.

We need two preliminary results on the derivative of Szàsz–Mirakjan operators applied to the special functions defined by (1.8): the first one deals with the derivative of the first order, the second one gives a recursive formula for the mth derivative.

To this end, we recall that, for a given real-valued function f on a real interval I, for every $x \in I$, $h \in \mathbb{R}$, $h \neq 0$ and $k \in \mathbb{N}$, the kth difference $\Delta_h^k f(x)$ of f with step h at the point x is defined by

$$\Delta_h^k f(x) := \sum_{r=0}^{k} (-1)^{r+k} \binom{k}{r} f(x + rh), \tag{2.1}$$

provided that $x + kh \in I$. As usual, we shall set $\Delta_h f(x) := \Delta_h^1 f(x)$.

It is also well known that, if there exist f', f'', ..., $f^{(n)}$ and $f^{(n)} \geqslant 0$, then $\Delta_h^n f(x) \geqslant 0$ for every $x \in I$ and $h \geqslant 0$ such that $x + nh \in I$.

Lemma 2.5. *Let us consider a function $\varphi \in E_\infty \cap C^{(1)}([0, +\infty[)$ such that $\varphi' \in E_\infty$. Then, for all $x \geqslant 0$, $n \geqslant 1$ and $p = 0, 1, ..., n$ one gets*

$$\frac{d}{dx}(M_p(\varphi_{n, p, x}))(x) = pM_p((\Delta_{1/n}\varphi)_{n, p, x})(x) + \left(1 - \frac{p}{n}\right) M_p(\varphi'_{n, p, x})(x).$$

Proof. By virtue of formula (1.7) one has

$$\frac{d}{dx}(M_p(\varphi_{n,\,p,\,x}))(x)$$

$$= pe^{-px}\sum_{h=0}^{\infty}\frac{(px)^h}{h!}\left[\varphi_{n,\,p,\,x}\left(\frac{h+1}{p}\right)\varphi_{n,\,p,\,x}\left(\frac{h}{p}\right)\right]$$

$$+\left(1-\frac{p}{n}\right)M_p(\varphi'_{n,\,p,\,x})(x)$$

$$= pe^{-px}\sum_{h=0}^{\infty}\frac{(px)^h}{h!}((\Delta_{1/n}\varphi)_{n,\,p,\,x})\left(\frac{h}{p}\right)+\left(1-\frac{p}{n}\right)M_p(\varphi'_{n,\,p,\,x})(x)$$

$$= pM_p((\Delta^1_{1/n}\varphi)_{n,\,p,\,x})(x)+\left(1-\frac{p}{n}\right)M_p(\varphi'_{n,\,p,\,x})(x). \quad \blacksquare$$

PROPOSITION 2.6. *Let us consider a function $f\in E_\infty\cap C^{(m)}([0,+\infty[)$ $(m\geqslant 1)$ such that $f',\dots,f^{(m)}\in E_\infty$.*
Then, for all $x\geqslant 0$, $n\geqslant 1$ and $p=0,1,\dots,n$ one gets

$$\frac{d^m}{dx^m}(M_p(f_{n,\,p,\,x}))(x)=\sum_{j=0}^{\infty}\binom{m}{j}p^j\left(1-\frac{p}{n}\right)^{m-j}M_p((\Delta^j_{1/n}f^{(m-j)})_{n,\,p,\,x})(x).$$

Proof. We shall prove the previous formula by induction on m.
Clearly, it holds true for $m=1$ in virtue of Lemma 2.5.
Suppose now that the above formula holds for a fixed $m\in\mathbb{N}$. Then, applying Lemma 2.5 to the function $\Delta^j_{1/n}f^{(m-j)}$ we obtain

$$\frac{d^{m+1}}{dx^{m+1}}(M_p(f_{n,\,p,\,x}))(x)$$

$$=\sum_{j=0}^{m}\binom{m}{j}p^j\left(1-\frac{p}{n}\right)^{m-j}\frac{d}{dx}(M_p((\Delta^j_{1/n}f^{(m-j)})_{n,\,p,\,x}))(x)$$

$$=\sum_{j=1}^{m+1}\binom{m}{j-1}p^j\left(1-\frac{p}{n}\right)^{m+1-j}M_p((\Delta^j_{1/n}f^{(m+1-j)})_{n,\,p,\,x})(x)$$

$$+\sum_{j=0}^{m}\binom{m}{j}p^j\left(1-\frac{p}{n}\right)^{m+1-j}M_p((\Delta^j_{1/n}f^{(m+1-j)})_{n,\,p,\,x})(x)$$

$$= \sum_{j=1}^{m} \left[\binom{m}{j-1} + \binom{m}{j} \right] p^j \left(1 - \frac{p}{n}\right)^{m+1-j} M_p((\Delta_{1/n}^j f^{(m+1-j)})_{n,p,x})(x)$$

$$+ p^{m+1} M_p((\Delta_{1/n}^{m+1} f)_{n,p,x})(x) + \left(1 - \frac{p}{n}\right)^{m+1} M_p((f^{(m+1)})_{n,p,x})(x)$$

$$= \sum_{j=0}^{m+1} \binom{m+1}{j} p^j \left(1 - \frac{p}{n}\right)^{m+1-j} M_p((\Delta_{1/n}^j f^{(m+1-j)})_{n,p,x})(x). \qquad \blacksquare$$

For each $m \in \mathbb{N}$, $m \geqslant 1$, we shall denote by $E_{\infty,+}^{(m)}$ the set of all functions $f \in E_\infty \cap C^{(m)}([0, +\infty[)$ such that

$$f', ..., f^{(m)} \in E_\infty \qquad \text{and} \qquad f^{(m)} \geqslant 0. \tag{2.2}$$

We also set

$$E_{\infty,+}^{(\infty)} := \bigcap_{m \geqslant 1} E_{\infty,+}^{(m)}. \tag{2.3}$$

Taking the above definition and the above result into account, an application of formula (1.9) gives immediately the following result.

THEOREM 2.7. *Suppose that the function λ is constant. Then, for each $n \geqslant 1$ and $m \geqslant 1$ we have*

$$M_{n,\lambda}(E_{\infty,+}^{(m)}) \subset E_{\infty,+}^{(m)} \qquad \textit{and} \qquad M_{n,\lambda}(E_{\infty,+}^{(\infty)}) \subset E_{\infty,+}^{(\infty)}.$$

3. PRESERVATION OF HÖLDER CONTINUITY

This section is devoted to the study of the preservation of the class of Hölder continuous functions. Here we present two results, the first of which contains also a regularity result concerning the semigroup $(T_\lambda(t))_{t \geqslant 0}$.

We shall denote by $Lip_K\, \alpha$ ($K \geqslant 0$, $\alpha \in]0, 1]$) the class of all functions $f \in C([0, +\infty[)$ such that

$$|f(y) - f(x)| \leqslant K\,|y - x|^\alpha \qquad (x, y \geqslant 0). \tag{3.1}$$

Note that $Lip_K\, \alpha \subset W_2^0$.

In order to show the first regularity result, we shall need the next general lemma essentially due to I. Rasa (unpublished paper).

LEMMA 3.1. *Let I be an interval of $\mathbb{R}$ and denote by $F(I)$ the linear space of all real-valued functions defined on I.*

Let E be a subspace of $F(I)$ containing the function e_1 defined by $e_1(t) := t$ $(t \geqslant 0)$. Consider a linear operator $L: E \to F(I)$ which maps increasing functions of E into increasing functions. Moreover, suppose that there exist two constants $c_0, c_1 \in \mathbb{R}$, $c_1 > 0$, such that $L(e_1) = c_1 e_1 + c_0 \mathbf{1}$, where $\mathbf{1}$ denotes the constant function of constant value 1.

Then

$$L(Lip_K\, 1 \cap E) \subset Lip_{c_1 K}\, 1.$$

Having the above lemma and Proposition 2.2 at our disposal, we shall easily obtain the following result.

PROPOSITION 3.2. *Assume that the function λ is constant. Then we have*

$$M_{n,\lambda}(Lip_K\, 1) \subset Lip_K\, 1 \qquad (n \geqslant 1).$$

Consequently,

$$T_\lambda(t)(Lip_K\, 1) \subset Lip_K\, 1 \qquad (t \geqslant 0).$$

Proof. Since $M_{n,\lambda}(e_1) = e_1$, the first assertion directly follows from Proposition 2.2 and Lemma 3.1.

In particular, one obtains the second inclusion from the representation formula (1.21). ∎

In a more general case, i.e., when α is not necessarily equal to 1, we can prove the following

THEOREM 3.3. *Assume that the function λ is constant. Then for every $\alpha \in\,]0, 1]$ we have*

$$M_{n,\lambda}(Lip_K\, \alpha) \subset Lip_{K(1 + B_\pi(e_n)(\lambda))}\, \alpha \qquad (n \geqslant 1).$$

where B_n denotes the nth Bernstein polynomial on $[0, 1]$, and e_α is the function defined by $e_\alpha(t) := t^\alpha$ $(t \geqslant 0)$.

Proof. Let $f \in Lip_K\, \alpha$ and fix $n \geqslant 1$ and $x, y \geqslant 0$.

Taking formula (1.9) into account we get

$$\begin{aligned} &|M_{n,\lambda}(f)(y) - M_{n,\lambda}(f)(x)| \\ &\qquad \leqslant \sum_{p=0}^{n} \binom{n}{p} \lambda^p (1-\lambda)^{n-p}\, |M_p(f_{n,p,y})(y) - M_p(f_{n,p,x})(y)| \\ &\qquad\quad + \sum_{p=0}^{n} \binom{n}{p} \lambda^p (1-\lambda)^{n-p}\, |M_p(f_{n,p,x})(y) - M_p(f_{n,p,x})(x)|. \qquad (1) \end{aligned}$$

Observe that, for every $p=0, 1, \ldots, n$, by using formulas (1.7), (1.8) and the hypothesis on f one gets

$$\begin{aligned}|M_p(f_{n,p,y})(y)-M_p(f_{n,p,x})(y)|\\ &\leqslant e^{-py}\sum_{h=0}^{\infty}\frac{(py)^h}{h!}\left|f\left(\frac{h}{n}+\left(1-\frac{p}{n}\right)y\right)-f\left(\frac{h}{n}+\left(1-\frac{p}{n}\right)x\right)\right|\\ &\leqslant Ke^{-py}\sum_{h=0}^{\infty}\frac{(py)^h}{h!}\left|\left(1-\frac{p}{n}\right)(y-x)\right|^{\alpha}\leqslant K\,|y-x|^{\alpha}. \qquad (2)\end{aligned}$$

On the other hand, $f\in Lip_K\,\alpha$ implies

$$f_{n,p,x}\in Lip_{K(p/n)^{\alpha}}\,\alpha \qquad (p=0, 1, \ldots, n). \qquad (3)$$

Now we recall that, in virtue of a result of Khan and Peters [7, Theorem 2 and Example 3.2], each Szàsz–Mirakjan operator M_p preserves Lipschitz class $Lip_{\bar{K}}\,\bar{\alpha}$ ($\bar{K}\geqslant 0$, $\bar{\alpha}\in\,]0, 1]$). In particular, from (3) it follows that

$$|M_p(f_{n,p,x})(y)-M_p(f_{n,p,x})(x)|\leqslant K\left(\frac{p}{n}\right)^{\alpha}|y-x|^{\alpha}. \qquad (4)$$

Consequently, inserting formulas (2) and (4) into (1), we obtain

$$\begin{aligned}|M_{n,\lambda}(f)(y)-M_{n,\lambda}(f)(x)| &\leqslant K\left[1+\sum_{p=0}^{n}\binom{n}{p}\lambda^p(1-\lambda)^{n-p}\left(\frac{p}{n}\right)^{\alpha}\right]|y-x|^{\alpha}\\ &=K[1+B_n(e_{\alpha})(\lambda)]\,|y-x|^{\alpha}. \qquad (5)\end{aligned}$$

So, the theorem is completely proved. ∎

Remark 3.4. From formula (5) of the above proof it follows that

$$M_{n,\lambda}(Lip_K\,\alpha)\subset Lip_{2K}\,\alpha \qquad (n\geqslant 1).$$

Remark 3.5. If the function λ is not constant, then, in general, $M_{n,\lambda}$ does not map the class $Lip_K\,\alpha$ into the class $Lip_{2K}\,\alpha$ for all $n\geqslant 1$ and, consequently, the statements of Proposition 3.2 and Theorem 3.3 do not hold true.

Indeed, if we consider the function $f\in Lip_1\,1$ defined by $f(x):=e^{-x}$ ($x\geqslant 0$), then $M_{1,\lambda}(f)(x)=e^{-x}\{1+\lambda(x)[e^{x/e}-1]\}$ (see [3, proof of Theorem 2.3 (1)]).

Choose now a function $\lambda \in C([0, +\infty[)$ such that

(i) $0 \leqslant \lambda(x) \leqslant 1$ $(x \geqslant 0)$;

(ii) λ is differentiable in an interval I_{x_0} containing x_0, where $x_0 := e \ln(e/(e-1))$;

(iii) $\lambda'(x_0) < -\lambda_0$, where $\lambda_0 := (e-1)[2(e/(e-1))^e - 1]$ (for example, the function λ defined by

$$\lambda(x) := \begin{cases} \alpha, & 0 \leqslant x \leqslant x_0 - \delta^2, \\ \alpha - \sqrt{x - (x_0 - \delta^2)}, & x_0 - \delta^2 \leqslant x \leqslant x_0 - \delta^2 + \alpha^2, \\ 0, & x \geqslant x_0 - \delta^2 + \alpha^2, \end{cases}$$

with $0 < \delta < \alpha < 1$ and $\delta < 1/2\lambda_0$).

It is easy to see that there exist a non empty set $J_{x_0} \subset I_{x_0}$ containing x_0 such that

$$\frac{d}{dx}(M_{1,\lambda}(f))(x) < -2 \qquad \text{for all} \quad x, y \in J_{x_0}.$$

Consequently,

$$|M_{1,\lambda}(f)(y) - M_{1,\lambda}(f)(x)| > 2\,|y - x| \qquad \text{for all} \quad x, y \in J_{x_0}$$

and, hence, $M_{1,\lambda}(f) \notin Lip_2 1$.

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